DCK-003-1164003 Seat No. ______

M. Sc. (Sem. IV) Examination

July - 2022

Mathematics : CMT-4003

(Number Theory - II)

Faculty Code : 003
Subject Code : 1164003

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) There are five questions.
(2) All the questions are compulsory.
(3) Each question carries 14 marks.

1 Do as directed : (Answer any **seven**) **14**

(a) Find out the values of r_1, r_2, r_3 of $\langle 0, 2, 2, 2, 2, 2 \rangle$.

(b) Find the period of $\sqrt{18}$.

(c) Find four positive integers for which $1 + 2 + 3 + 4 \ldots$ $\ldots + n$ is a perfect square.

(d) Find the general solution if any of the equation $131x + 211y = 2$.

(e) If (x, y, z) is a Pythagorean Triplet then show that, g.c.d (y, z) = g.c.d (x, z).

(f) Write down, any two solution of $3x + 5y = 8$.

(g) Prove that, the g.c.d. (x, y) where $\frac{x}{y}$ farey fraction of the n^{th} row is 1.

(h) Find the value of $\langle 0, 3, 3, 3, 3, \ldots, \ldots \rangle$ and $\langle -2, 1, 2, 6, 14 \rangle$.

(i) Express the numbers $\sqrt{7} + 2$ and $\sqrt{3} - 1$ in continued fraction expansion.

(j) Write the statement of Hurwitz's Inequality for $x, y > 0$ for farey fractions.

2 Answer any two of the following : **14**

(1) Solve the linear Diophantine equation $172x+20y=1000$ by usual method.

(2) Prove that, $x^2-143y^2+1=0$ has no solution in integers.

(3) If p, q is a positive solution of $x^2-ny^2=1$, then $\frac{p}{q}$ is a convergent of the continued fraction of $\sqrt{n}$.

3 Answer the following : **14**

(1) Suppose $C_n x^n + C_{n-1}x^{n-1} + \ldots\ldots + C_0$ is a polynomial with integer co-efficient and $\frac{s}{t}$ is a rational number with t is a positive and $(s, t) = 1$. Let $\frac{s}{t}$ be a root of this polynomial. Prove that, s divide to C_0 and t divide to C_n. Hence, deduce that, if a is an integer and $x^n = a$ has a rational root then it must be an integer.

(2) (i) Suppose r and s are positive integer such that $r > s \geq 1$, $(r, s) = 1$ and r is even then s is odd and vice-versa then prove that, the triplet (x, y, z) is a Primitive Pythagorean triplet where $x = r^2 - s^2$, $y = 2rs$ and $z = r^2 + s^2$.

(ii) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots \ldots \ldots + a_1 x + a_0$ be a polynomial with integer coefficient with degree n and if $\frac{a}{b}$ satisfies $f(x)$ then prove that, $b \mid a_n$ and $a \mid a_0$ provided $(a, b) = 1$ and $b \geq 0$.

OR

3 Answer the following : **14**

(1) Prove that, the value $f(x) = x^4 + x^3 + x^2 + x + 1$ is a perfect square for $x = -1, 0$ and 3 and for some values of x, $f(x)$ is not a perfect square.

(2) Prove that, if x is a rational number then its finite simple continued fraction expansion is always unique provided the last term is not equals to 1.

4 Answer the following : **14**

(1) Suppose (x_1, y_1) is a smallest positive solution of $x^2 - dy^2 = 1$ then prove that :

(i) $(x_1 + \sqrt{d}y_1)^n$ is a solution of it for $n \geq 1$.

(ii) Every positive solution is of the form $(x_1 + \sqrt{d}y_1)^n$ for some n.

(2) (i) If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive farey fractions in the n^{th} row then show that, $|ad - bc| = 1$.

(ii) Find the general solution of $4x + 27y = 22$.

5 Answer any **two** of the following : **14**

(1) Show that, the equation $x^4 = z^2 - y^4$ has no solution in integers.

(2) If $a_0, a_1, a_2, ..., a_n, ...$ are integers with $a_i \geq 1; \forall\, i \geq 1$ then the number θ is and irrational number if and only if the expansion $\langle a_0, a_1, a_2, ..., a_n, ...\rangle$ is infinite.

(3) Suppose θ is a quadratic irrational such that $\theta > 1$ and $-1 < \theta' < 0$. Prove that its periodic continued fraction expansion is pureluy periodic. Confirm the result for $\frac{\sqrt{3}+1}{2}$.

(4) Find first four positive solution of $x^2 - 19y^2 = 1$.